Thanks, Louis, You Were Right!! 100 Years of Heavy Tails—The Hypothesis That Won't Go Away

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Preamble

I wrote the first draft of this at the turn of the present century, very near the actual 100th anniversary of the publication of Louis Bachelier's thesis. Little did I know that the refinement of the paper, review, and publication process would cover years in which tumultuous events would draw the subject of finance and real estate to dramatic and near-disastrous proximity. As this article approaches publication the arcane subjects of extreme value and heavy tails, once dwelling in the obscurity of academia, have emerged to enter the lore of the popular press. Today the idea of an ordered, normal world is vigorously debated at all levels of society. Whether this piece offers insight or merely adds to the cacophony I leave to the reader to decide.

—RJB

Introduction

It must have been no small satisfaction to Louis Bachelier[1] when the great French mathematician Paul Levy wrote him a letter of apology for misinterpreting and ridiculing the work of his dissertation some 30 years earlier.[2] If the two were alive today, Levy could collect a number of signatures also thanking Louis.

Although apparently unaware of Bachelier's work, Einstein's theory of relativity is based on it. The Ito calculus that lead to the Black-Scholes model is derived from it. Our ability to transfer data over telephone lines is based on it. Orange County, Barings Bank, and Long Term Capital Management all paid a high price for ignoring it.[3] In the real world, anyone with a bumper sticker that proclaims "S* * * Happens" knows about it. The simple fact is that we live in a world where extreme values occur, create heavy tailed distributional outcomes and, on occasion, profoundly influence our lives.

The formal name for distributions with heavy tails is "stable" or "Stable-Paretian (SP)" because they approach a Pareto distribution.[4] Why, then, does academic real estate resist it so mightily? Is it simply the defense of the dominant theory? Is it turf protection among tenured faculty who "know what they know" and shun change? Or is it traceable to the practical difficulties of implementation? These all constitute the usual suspects that block establishment and acceptance of a new paradigm.

Rather than pick one of these, this paper

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seeks to review the present condition of heavy tail thought, acknowledge its weaknesses, highlight some recent relief from technological barriers that offer a light at the end of the implementation tunnel, and point the way to a new direction that accommodates reality without sacrificing elegance.

Important Developments of the Last Half of the 20th Century

The work of Mandlebrot[5] in commodity prices and Fama[6] in stock prices was heroic. They had to deal with data in written form and illustrate their findings on Log-Log paper with a #2 lead pencil. Roll[7] used a computer to find the heaviest of heavy tails in interest rates. All of this was done without knowing if stable distributions were unimodal, something proved by Yamazato,[8] and with no way to cross the important $\alpha = 1$ parameter value, which McCulloch[9] later resolved.

Using McCulloch's tables, and after a three-year struggle to get published, Young and Graff[10] showed that heavy tails are pervasive in Tier III real estate.[11] Brown[12] extended that to Tier II real estate with similar findings. Very recently, Nolan[13] made possible the estimation of stable parameters across the entire parameter space using numerical integration in a Fortran environment. McCulloch[14] used real estate data to show that beta estimates are substantially misstated when regression is performed in the presence of stable disturbances.

Stable regression has been implemented in MatLab and SPlus and will soon be available on the internet. Due to Rimmer,[15] today it is possible to generate stable random numbers, estimate parameters and confidence intervals, create a pdf and CDF for a stable distribution and examine tail behavior in a wide range of ways over the internet. What was once a tedious and unpleasant task fraught with theoretical problems is now accessible to virtually anyone with a computer.

Still, many in academic real estate are not persuaded. This, in the face of unassailable evidence that the assumption of normality understates risk in *every case* where heavy tails exist, simply because the normal distribution (which is just one of the family of an infinite number of stable distributions) acts as a blinder, *precluding* the consideration of extreme values. We hope this is neither institutionalized myopia nor psychological arterial sclerosis.[16] The purpose of this paper is to examine what valid reasons may exist and suggest a path out of the thicket.

Why Modern Portfolio Theory[17] is so Enduring

Classical philosophic thought strives for the maximum generality. Everyone salutes the theory with wide application in a universal sense (think planetary heliocentric theory here). In social sciences, where mathematics are co-opted for sustenance, one hopes that any theory finds support along the entire real number line $(-\infty, \infty)$ and is buttressed by a closed form analytic solution. Theories lacking these properties are understandably suspect. The stable hypothesis relies on numerically intensive solutions and must often resort to particular bounded areas of parameter space for a solution, either of which has the smell of *ad hoc* to the serious philosopher.

Reasonable questions include:

- Are we to rely on charming rogues with computers for our theories?
- Can a practical application with intuitive merit rise to the level of a theory?

- Will the estrangement of theory and practice widen or narrow at this critical time?

Markowitz' great insight was that risk could be managed independently of return.[18] He showed us how to double the number of investment decisions parameters and his proof, thanks to some strong assumptions, has a closed form solution. Key to his contribution is the position that only one parameter, variance, describes risk. In 1952 when *no* parameters addressed risk this was simply a big deal.

Finance as an academic discipline was carved out from microeconomics by relaxing the assumption of perfect certainty. It may be that academic real estate will find its birthright by relaxing the assumption of normality. Valid precedence for this, set forth by a pioneer in mathematical statistics, is Mandlebrot's suggestion of Second Stage Indeterminism.[19] If real estate is—as it appears to be—clearly out of reach of the First Stage Indeterminism of Modern Portfolio Theory (MPT), real estate as an academic study is truly, and finally, distinguished from finance.

With any paradigm shift there are benefits and costs. We will address those below.

The 21st Century Case for the Stable Hypothesis

A general stable distribution requires four parameters to describe it:

- The index of stability or characteristic exponent $\alpha \in (0,2]$,
- The skewness parameter $\beta \in [-1,1]$,
- The scale parameter $\gamma > 0$,
- The location parameter $\delta \in R$.

However, before we embark on this journey it is useful to accept a few things. In no special order the warts on the stable hypothesis that remain are:

1. *There is no closed form solution* that results in a pdf for any but three (Levy, Cauchy, and normal) stable distributions. Two of these (Levy and Cauchy) are not useful for financial applications.[20] Thus, the relaxation of the assumption of normality means that all pdfs will be produced in the transform space of the characteristic function. While this was once a vexatious undertaking, with a gigabyte of RAM on everyone's desk, it is no longer really a problem.[21]

2. *The parameters of the stable characteristic function are not independent.* Although stable laws are additive and shape retaining under linear transformation (two useful properties when one wants to "normalize" the scale and location parameters), all four parameters must be considered together.

3. *There is at present no way to prove that a data set is stable.* Very reliable tests (such as Jarque-Bera) exist to prove that data are nonnormal. There is a test for infinite variance[22] and graphical tests[23] to *indicate* stable behavior.

4. *The stable ch. f. may be described by multiple parameterizations.* There are at least four and perhaps 11 of these (Nolan, in press), each offering different benefits. Perhaps all but two are the domain of the theoretical mathematician. Of these two, one is better for theoretical work as it is continuous in all four parameters and the other is better for applied work as one of the parameters, δ, is the mean.

5. *For nonnormal stable distributions, in the limit no higher moments exist.* No variance means no covariance, no efficient sets or frontier with all the repercussions that entails for the Capital Asset Pricing Model (CAPM).[24] All is not lost, however. Fama and Miller observed that using bounded sets one always has a variance and the cure for heavy tails was merely more assets than indicated by the MPT under the assumption of normality. However, α—the index of tail thickness—is an exponent. Thus as it declines away from the normal (the case of $\alpha = 2$) an exponentially growing, and for real estate an unrealistically high, number of assets is required to maintain a constant level of risk.

6. *Analysis requires large datasets.* While it is possible to work with stable laws where n<1000, the result is wide confidence intervals. Ideally, work with stable laws starts with data sets >2000, often a difficult bar for real estate to clear.

7. Lastly, and perhaps the most fatal for applying the finance paradigm to real estate, is that while efficient set mathematics can be performed on finite sets, *one really does not know what it means to place a four parameter model in two parameter space* just to produce a "frontier," efficient or not.

Perhaps the only useful result of this exercise is to show that the stable frontier *always* plots to the right of the normal equivalent, underwriting the notion that the normal assumption understates risk. But basic statistics presume a sample is drawn from a larger, unobservable population. If assumptions are made about that population, those assumptions are usually about its limiting distribution. To presume that the population is stable and then to perform efficient set mathematics on a sample from it using variance and covariance as if the population from which it was drawn were normal is, at best, a kludge.

Toward A New and Better Paradigm for Real Estate

As Friedman[25] famously reminded us, the importance of a theory is not that it always be correct but whether it helps us make useful predictions. With that in mind, if the preponderance of the evidence is that real estate has heavy tails, and that the tails are too heavy to be ignored,[26] *one now faces the unappetizing choice of using an elegant paradigm that always provides a solution but is always wrong or using a paradigm that occasionally does not provide a solution at all but when it does is more accurate. Such is the dilemma facing academic real estate today.*

The most radical position one might take is that it is and always was misguided to form real estate portfolios at all. Young and Graff[27] make the point that site specific risk is so important that failing to fully consider it as part of the acquisition process can be the perverse result of expending effort to form portfolios to lower risk when risk either can't be lowered or is lowered by a very little relative to the time (and related management difficulties) associated with forming a portfolio. Brown[28] takes an even more extreme counter position for noninstitutional size property in private hands claiming that real estate is the domain of the individual entrepreneur whose primary goal is and should be not to diversify at all but to put all his eggs in one basket and ***watch that basket!!***

Somewhere in the middle lies the truth. Diversification clearly has benefits, especially

in a mixed asset portfolio. Whether strict Markowitz gains from diversification are available is in doubt. However, no one should doubt the simple street wisdom that preceded Markowitz: Spreading the risk around makes good sense. The question facing academia is similar to that facing practitioners: How much more effort should be expended further wearing the MPT rut smoother and shinier when it is not likely to take us any deeper?

The beginning point of new thought, alas, is not new. Skewness and kurtosis have existed for a long time. They just are assumed away for tractability in most theoretical settings where the normal assumption is convenient. Suppose we argue that the relaxation of the assumption of normality permits *tripling* the number of parameters used to describe risk.[29] Is this a new definition of risk? Not really, we are just *permitting* the view to include extreme values concentrated in the either tail.[30] And why shouldn't we? Real players with real money cannot afford to assume away reality, especially a particularly dangerous one. The Street sees numerical methods not as leprosy but as a practical solution to a problem, something they resort to every day. If repeated trials produce better predictions, practitioners are little concerned of the fine points of a theory's elegance or lack thereof.

Fifty years after MPT it may be ludicrous to try to explain the investing world with two numbers. It may be only half as ludicrous to try to do so with four. However, the wider view of risk will not be denied. Perhaps the best use of stable laws is in looking not at groups of assets but entire markets. If the Milwaukee apartment market has an α of 1.5 and a β of .7 does it offer *more opportunity* than the Omaha market with an α of 1.8 and a zero β? Is it too much of a stretch to view regions as collections of assets that may not be possible to acquire as an entire portfolio but attractive or unattractive target markets in which to expose oneself to a certain level of risky activity?

Regardless, *we now know that MPT in its usual incarnations does not work for real estate as we once thought it might. Where we go from here becomes the interesting academic real estate challenge of the new century.*

NOTES:

[1]Bachelier, L. J. B. A. Theorie de la speculation. Paris: Gauthier-Villars (1900).

[2]Levy, P. Theorie de l'addition des variables aleatoires. Paris, France: Gauthier-Villars (1937); Levy, Paul. Calcul des probabilities. Paris: Gauthier-Villars (1925).

[3]These are the spectacular, headline-making failures. One can only presume that smaller, less noticeable errors occur at all levels.

[4]Pareto, V. Cours d'Economie Politique. Lausanne, Switzerland (1897).

[5]Mandelbrot, Benoit. The Variation of Certain Speculative Prices. Journal of Business. 1963 Oct; 36(4):394-419.

[6]Fama, Eugene F. The Behavior of Stock-Market Prices. Journal of Business. 1965 Jan; 38(1):34-105.

[7]Roll, Richard. The Behavior of Interest Rates; The Application of the Efficient Market Hypothesis to U.S. Treasury Bills. New York: Basic Books (1970).

[8]Yamazato, M. Unimodality of infinitely divisible distributions of class L. Ann. Probability. 1978; 6:523-531.

[9]McCulloch, J. Huston. Simple Consistent Estimators of Stable Distribution Parameters. Communications in Statistics: Simulation and Computation. 1986; 15(4):1109-1136.

[10]Young, Michael S. and Graff, Richard A. Real Estate is Not Normal: A Fresh Look at Real Estate Return Distributions. Journal of Real Estate Finance and Economics. 1995 May; 10(3):225-259 (1995).

[11]Brown (2000) distinguishes real estate by size where Tier I is the very small (1-4 owner occupied and the like), Tier III is the very large, and Tier II is property in the middle, typically privately owned but too large for owner consumption and too small to qualify for institutional interest.

[12]Brown, Roger J. Return Distribution of Private

Real Estate Investments. University Park, PA: The Pennsylvania State University; 2000 May; Brown, Roger J. Risk and Private Real Estate Investments. Journal of Real Estate Portfolio Management. 2004 Summer; 10(2):113.

[13]Nolan, John P. Maximum Likelihood Estimation and Diagnostics for Stable Distributions, [Working Paper] (1998).

[14]McCulloch, J. Huston. Linear Regression with Stable Disturbances. Adler, Robert J.; Feldman, Raisa E., and Taqqu, Murad S., Editors. A Practical Guide to Heavy Tails: Statistical Techniques and Applications. Boston, MA: Birkhauser; 1998; pp. 359-376.

[15]Rimmer, Robert H. M. D. Calculation of Stable Distributions with Mathematica [Web Page]. 2004. Available at: www.mathestate.com http://www.npgcable.com/%7Errimmer/StableCalculation/.

[16]Hardening of the attitudes.

[17]Modern Portfolio Theory (MPT) holds that one can achieve the same return at a lower risk by judiciously combining assets. A common metaphor is the umbrellas-and-bathing-suits approach to investing.

[18]Markowitz, Harry M. Portfolio Selection. Journal of Finance. 1952 Mar; 3:77-91. Mean is mathematically independent of variance.

[19]Mandelbrot, Benoit B. Towards a Second Stage of Indeterminism in Science. Interdisciplinary Science Reviews. 1987; 12(2):117-127.

[20]Fortunately, most financial data exhibit α values of [1,2), obviating the need to use the other two which have α values less than 1, where the stable distribution does not even have a mean.

[21]Although Excel will not suffice. One needs powerful symbolic computing software and large RAM for numerically intensive solutions.

[22]Granger, Clive W. J. and Orr, Daniel. "Infinite Variance" and Research Strategy in Time Series Analysis. Journal of the American Statistical Association. 1972 Jun; 67(338):275-285.

[23]DuMouchel, W. H. Stable Distributions in Statistical Inference: Yale University (1971); Rimmer, Robert H. M. D. Calculation of Stable Distributions with Mathematica [Web Page]. 2004. Available at: www.mathestate.com http://www.npgcable.com/%7Errimmer/StableCalculation/.

[24]The Capital Asset Pricing Model (CAPM) holds that unless return is appropriately matched with risk an asset will not be held in the portfolio. Thus, the asset must be correctly "priced" by the market in order to sell and that price must include an accurate measure of its risk. In 1990 Harry Markowitz, William Sharp and Merton Miller shared the Nobel Prize in Economics for their contributions of MPT and CAPM.

[25]Friedman, Milton. Essays in Positive Economics. Chicago, IL: The University of Chicago Press (1953).

[26]Finance authors found α to be above 1.7 and called that close enough to normal for their purposes. Young and Graff (1995) and Brown (2004) found α to be in the 1.5 range. As α is an exponent, this is a nontrivial difference. Wildly different outcomes arise when tails are that heavy.

[27]Young, Michael S. and Graff, Richard A. Real Estate is Not Normal: A Fresh Look at Real Estate Return Distributions. Journal of Real Estate Finance and Economics. 1995 May; 10(3):225-259 (1995).

[28]Brown, Roger J. Risk and Private Real Estate Investments. Journal of Real Estate Portfolio Management. 2004 Summer; 10(2):113.

[29]Using δ as the mean, the three remaining parameters of the stable ch. f., α, β, γ, all dictate nonlocation (shape) features of the distribution.

[30]No small matter here is the appeal to the Generalized Central Limit Theorem, the notion of additivity and shape preserving under linear transformation, properties available in the robust SP distribution family and not for other common distributions like Student t.